Club Formation Games with Farsighted Agents

Frank H. Page, Jr.
Department of Finance
University of Alabama
Tuscaloosa, AL 35487
fpage@cba.ua.edu

Myrna H. Wooders*
Department of Economics
Vanderbilt University
Nashville, TN 37235
m.wooders@vanderbilt.edu

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Abstract

Modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if agents behave farsightedly in choosing their club memberships. Using the farsighted core as our stability notion, we show that if agents' payoffs are single-peaked and agents agree on the peak club size (i.e., agents agree on the optimal club size) and if there sufficiently many clubs to allow for the partition of agents into clubs of optimal size, then a necessary *and* sufficient condition for the farsighted core to be nonempty is that agents who end up in smaller-than-optimal size clubs have no incentive to switch their memberships to already existing clubs of optimal size. In contrast, we show via an example that if there are too few clubs relative to the number of agents, then the farsighted core may be empty. Contrary to prior results in the literature involving myopic behavior, our example shows that overcrowding and farsightedness lead to instability in club formation.

1 Introduction

The study of club formation has a long history in economics going back to Buchanan (1965). Here we offer a new approach to the study of clubs. In particular, modeling club structures as bipartite networks, we formulate the problem of club formation as a game of network formation and identify those club networks that are stable if agents behave farsightedly in choosing their club memberships. Thus we bring together two strands of the literature: club theory[1] and the theory of social and economic networks initiated by Kirman (1983).

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*Also, Department of Economics, University of Warwick, Coventry CV4 7AL, UK.

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[1]See Demange and Wooders (2005), Part II for surveys of club theory from several perspectives.

Unlike the random graph theoretic approach taken by Kirman (1983), here we follow an approach similar to that taken by Jackson and Wolinsky (1996) in their study of networks and focus exclusively on strategic considerations in club network formation. The basic setup of our model is closely related to the model of Konishi, Le Breton and Weber (1997). They examine, however, free mobility equilibrium of a local public goods economy (an assignment of players to clubs, locations, or jurisdictions that partitions the population and has the property that no individual can gain by either moving to any other existing club, or creating his own club). The partition derived from the players' strategy choices is thus stable against unilateral deviations by individuals.[2]

In contrast to the prior literature on clubs, we allow strategic coalitional moves and permit agents to be farsighted.[3] Using the farsighted core introduced in Page and Wooders (2004) as our stability notion, we show that if agents' payoffs are single-peaked and agents agree on the club size at which payoffs peak (i.e., agents agree on the optimal club size) and if there are sufficiently many clubs (i.e., sufficiently many club types or club locations) to allow for the partition of agents into clubs of optimal size, then a necessary *and* sufficient condition for the farsighted core to be nonempty is that agents who end up in smaller-than-optimal size clubs (i.e., the left-over agents) have no incentive to switch their memberships to already existing clubs of optimal size.[4] We note that in this case, the outcome of farsighted behavior corresponds to outcomes of myopic behavior as in Arnold and Wooders (2005) and the set of outcomes in the farsighted core correspond to the 'Nash club equilibrium outcomes'.

The coincidence of outcomes of farsighted behavior and myopic behavior does not extend to all cases, however. We demonstrate via an example that if there are too few clubs relative to the number of agents so that on average clubs must be larger than optimal size, then the farsighted core may be empty. This emptiness problem is caused by the fact that farsighted agents, unlike myopic agents, might switch their club memberships to already overcrowded clubs, temporarily making themselves worse off, if in the end switching induces an out migration that makes them better off. We note that the Arnold and Wooders club formation model agents behave myopically in choosing their club memberships and will switch memberships if and only if switching makes them strictly better off next period. Thus, in their model, since agents are assumed to be unwilling to make themselves temporarily worse off, even if doing so induces payoff improving future out migrations, fewer membership defections are possible. As a result, Arnold and Wooders are able to show that

[2] In a similar set up, Conley and Konishi (2002) analyze *migration proof* equilibrium, which are stable only against *credible* deviations on the part of a coalition. A coalitional deviation to another jurisdiction is credible if no outsiders to the coalition will want to follow the deviators and, within the deviating group, no player can gain by a further deviation. Conley and Konishi consider only the case where the number of possible clubs is unconstraining.

[3] Our approach differs from the cooperative/price-taking approach in much of the literature on clubs (again see Part II of Demange and Wooders) in that coalitions behave strategically.

[4] Stated loosely, a club network is contained in the farsighted core if no group of agents has an incentive to alter their club memberships, taking into account club membership changes that might take place in the future.

even when there are too few clubs, a club structure in which all clubs are of nearly equal size is immune to coalitional defections. Using their terminology, Arnold and Wooders are able to show that if agents are myopic and if there is overcrowding, a Nash club equilibrium always exists. In contrast, our analysis suggests that in general overcrowding and farsightedness may lead to instability in club formation.

We shall proceed as follows. In Section 2, we introduce the notion of a club network and state the assumptions of our model. In Section 3, we define the farsighted dominance relation over the feasible set of club networks, and we define the farsighted path dominance relation. In Section 4, we define the abstract club network formation game with respect to the farsighted path dominance relation and we define the farsighted core of the club network formation game. Finally, in Section 4, we state our main result giving necessary and sufficient conditions for nonemptiness of the farsighted core for the case in which there are sufficiently many clubs.

2 Clubs Networks

We begin by introducing the notion of a club network. Using bipartite networks we are able to represent in a very compact and precise way the totality of any given club structure.

Let N be a finite set of agents consisting of two or more agents with typical element denoted by i, and let C be a finite set of club types - or alternatively, a set of club labels or club locations - with typical element denoted by c.

Definition 1 *(Club Networks)*
A club network g is a nonempty subset of $N \times C$ such that $(i, c) \in g$ if and only if agent i is a member of club c.

Given club network g,

$$g(c) := \{i \in N : (i, c) \in g\}$$

(i.e., the section of g at c) is the set of members of club c in network $g \subseteq N \times C$, while the set

$$g(i) := \{c \in C : (i, c) \in g\}$$

(i.e., the section of g at i) is the set of clubs to which agent i belongs in network $g \subseteq N \times C$.

Example 1 *To illustrate, suppose there are five agents $N = \{i_1, i_2, i_3, i_4, i_5\}$ and two clubs $C = \{c_1, c_2\}$. Further, suppose that c_1 denotes the chess club while c_2 denotes the fencing club. Club network g_0 depicted in Figure 1 represents one possible club* *structure given N and C.*

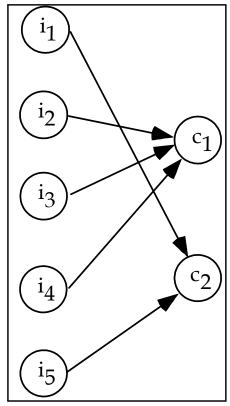

Figure 1: Club Network g_0

In club network g_0 the chess club has three members

$$g_0(c_1) = \{i_2, i_3, i_4\},$$

while the fencing club has two members

$$g_0(c_2) = \{i_1, i_5\}.$$

Note that in club network g_0 each agent is a member of one and only one club. Thus, for example

$$g_0(i_5) = \{c_2\},$$

that is, agent i_5 is a member of the fencing club, but is not a member of the chess club. Below we will formalize the single club membership property of this example in an assumption that we will maintain throughout the paper.

The collection of all club networks given N and C is given by the collection of all *nonempty* subsets of $N \times C$, denoted by $P(N \times C)$. We shall denote by $|g(c)|$ the number of members of club c (i.e., the club size) in network g and by $|g(i)|$ the number of clubs to which i belongs in network g. In Example 1, the chess club has three members, that is $|g_0(c_1)| = 3$, and agent i_5 belongs to one club - the fencing club - and thus $|g_0(i_5)| = 1$.

We shall maintain the following assumptions throughout:

A-1 (single club membership) The feasible set of club networks, $\mathbb{K} \subset P(N \times C)$, is given by

$$\mathbb{K} \subseteq \{g \in P(N \times C) : |g(i)| = 1 \text{ for all } i \in N\}.$$

Thus, in each feasible club network $g \in \mathbb{K}$ each agent is a member of one and only one club. Again note that club network g_0 in Example 1 satisfies the single club membership assumption [A-1]. Also note that under assumption [A-1] the collection $\{g(c) : c \in C\}$ forms a partition of the set of agents.

A-2 (identical payoff functions depending on club size) Agents have identical payoff functions, $u(\cdot)$, and payoffs are a function of club size only. In Example 1, agent i_5 is a member of the fencing club, that is, $g_0(i_5) = \{c_2\}$, and this club has a membership set given by

$$g_0(g_0(i_5)) := g_0^2(i_5) = \{i_1, i_5\}.$$

Thus, in network g_0 agent i_5 has a payoff given by

$$u(|g_0(g_0(i_5))|) = u(\left|g_0^2(i_5)\right|) = u(|\{i_1, i_5\}|) = u(2).$$

In general, given any club network g, $\left|g^2(i)\right|$ denotes the total number of club members in the club to which agent i belongs.

A-3 (single-peaked payoffs) There exists a club size s^* with $1 \leq s^* < |N|$ such that payoffs are increasing in club size up to club size s^* and decreasing thereafter.

A-4 (free mobility) Each agent can move freely and unilaterally from one club to another. This means that an agent can drop his membership in any given club and join any other club without bargaining with or seeking the permission of any agent or group of agents. In this sense our model of club formation as a game over club networks is noncooperative. The assumption of free mobility is quite common in models of noncooperative network formation (see, for example, Bala and Goyal (2000)), as well as in the club literature (see, for example, Demange (2005) and the references contained therein).

Example 2 *It is important to note that our assumptions do not rule out the possibility that some clubs have no members (i.e., are empty). Thus, in some feasible club networks $g \in \mathbb{K}$, it may be the case that $g(c) = \emptyset$ for some club type $c \in C$. If club c has no members, then $|g(c)| = |\emptyset| = 0$. Figure 2 depicts just such a situation.*

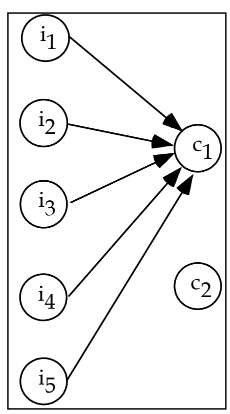

Figure 2: Club Network g_1

In moving from club network g_0 in Example 1 to club network g_1 above, agents i_1 and i_5 have freely and unilaterally dropped their memberships in the fencing club and joined the chess club. Thus, in club network g_1 the fencing club c_2 has no members.[5]

[5]While we assume that in moving from club network g_0 to club network g_1 agents i_1 and i_5 act

3 Dominance Relations Over Club Networks

Under the assumption of free mobility agents can alter any existing club network by simply switching their memberships. Such membership changes however can trigger further membership changes by other agents which in the end leave some or all of the agents who initially switched not better off and possibly worse off. Here we will assume that agents make their membership decisions taking into account the possibility of future membership changes by other agents - that is, we will assume that agents are farsighted and are concerned with the long run consequences of their immediate actions in choosing their club memberships. We begin by formalizing a notion of farsighted dominance. Then, using this farsighted dominance relation over club networks, we will identify club networks (i.e., club structures) that are farsightedly stable.

3.1 Farsighted Dominance

Throughout let S denote a *nonempty* subset of N.

Definition 2 *(Feasible Change and Improvement) Let g_0 and g_1 be two club networks in $\mathbb{K}$ $(g_0 \neq g_1)$.*

(1) (Feasible Change) *We say that agents $i \in S$ can feasibly change club network g_0 to club network g_1, denoted*

$$g_0 \underset{S}{\longrightarrow} g_1,$$

if the move from network g_0 to network g_1 only involves a change in club memberships by agents in S, leaving unchanged the memberships of agents outside group S, that is, if

$$\textit{if } g_0(i) = g_1(i) \textit{ for all agents } i \in N \backslash S \textit{ (i.e, } i \textit{ not contained in } S\textit{).}$$

(2) (Improvement) *We say that club network g_1 is an improvement over club network g_0 for agents $i \in S$, denoted*

$$g_1 \succ_S g_0,$$

$$\textit{if } u(|g_1^2(i)|) > u(|g_0^2(i)|) \textit{ for agents } i \in S.$$

(3) (Feasible Improvement) *We say that club network g_1 is a feasible improvement over club network g_0 for agents $i \in S$, denoted*

$$g_1 \vartriangleright_S g_0,$$

freely and unilaterally in switching their memberships, our model does not address the question of how agents i_1 and i_5 come to simultaneously switch their memberships, whether by communication and collusion or by serendipity. In order to formally address this question additional structure would have to be added to the current model. Page, Wooders, and Kamat (2005) make a start on addressing this question via the introduction of the supernetwork (i.e., a network of networks) in which the arcs represent coalitional moves and coalitional preferences (see also Page and Wooders (2004)).

$$\text{if } g_0 \underset{S}{\longrightarrow} g_1 \text{ and } g_1 \succ_S g_0.$$

(4) (Farsightedly Feasible Improvement) *We say that club network $g_* \in \mathbb{K}$ is a farsightedly feasible improvement over club network $g \in \mathbb{K}$ (or equivalently, we say that club network g_* farsightedly dominates club network g), denoted*

$$g_* \rhd\rhd g,$$

if there exists a finite sequence of club networks, $g_0, \ldots, g_n$, with $g := g_0$ and $g_* := g_n$, and a corresponding sequence of sets of agents, $S_1, \ldots, S_n$, such that for $k = 1, 2, \ldots, n$,

$$g_{k-1} \underset{S_k}{\longrightarrow} g_k \text{ and } g_n \succ_{S_k} g_{k-1}.$$

Thus, club network g_* is a farsighted feasible improvement over club network g if (i) there is a finite sequence of feasible changes in club networks starting with network g and ending with network g_*, and if (ii) payoffs

$$\left(u(g_*^2(i))\right)_{i \in N}$$

in *ending* club network g_* are such that for each k and for the agents in each coalition S_k, payoffs in the ending club network g_* are greater than the payoffs agents in S_k would have received in club network g_{k-1} (i.e., in the club network that agents in S_k changed) - that is, for each k

$$u(g_*^2(i)) := u(g_n^2(i)) > u(g_{k-1}^2(i)) \text{ for } i \in S_k.$$

The definition of farsighted feasible improvement above is a network rendition of Chwe's (1994) definition.

Example 3 *Suppose that there are seven agents and two clubs and that the optimal club size is three. Figure 3 depicts three feasible club networks, g_0, g_1, and g_2. Club network g_2 farsightedly dominates club network g_0.*

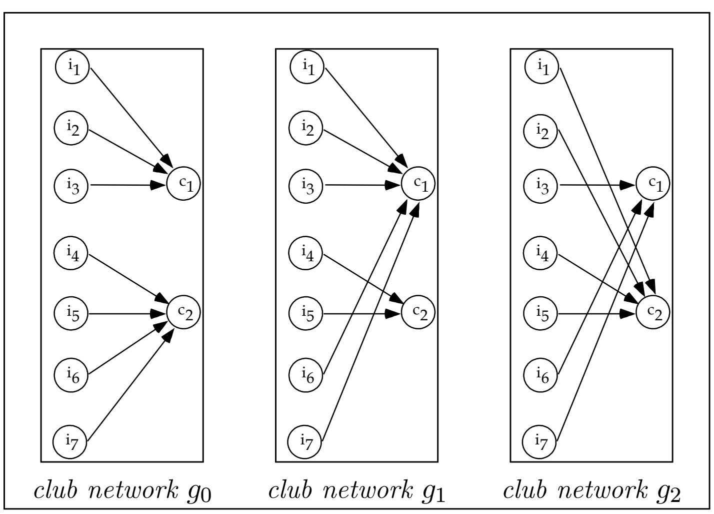

Figure 3: Three Possible Club Structures

To see this, consider the following sequence of moves. First, agents i_6 and i_7 switch their memberships from club c_2 to club c_1. This feasible move by agents i_6 and i_7 changes club network g_0 to club network g_1 and is denoted by

$$g_0 \underset{\{i_6,i_7\}}{\longrightarrow} g_1.$$

Second, agents i_1 and i_2 switch their memberships from club c_1 to club c_2. This feasible move by agents i_1 and i_2 changes club network g_1 to club network g_2 and is denoted by

$$g_1 \underset{\{i_1,i_2\}}{\longrightarrow} g_2.$$

Given an optimal club size of 3 and given the assumption of single-peaked payoffs, the initial moves by agents i_6 and i_7 makes them worse off.[6] *In particular, agents i_6 and i_7 start out in club c_2 in network g_0 with 4 members $\{i_4, i_5, i_6, i_7\}$ and payoffs given by*

$$u(\left|g_0^2(i_6)\right|) = u(\left|g_0^2(i_7)\right|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4),$$

and move to club c_1 creating a new club network g_1 in which club c_1 has 5 members $\{i_1, i_2, i_3, i_6, i_7\}$. As a result, agents i_6 and i_7 are made worse off with payoffs given by

$$u(\left|g_1^2(i_6)\right|) = u(\left|g_1^2(i_7)\right|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5).$$

However, due to the second round of moves by agents i_1 and i_2, agents i_6 and i_7 end up in a smaller club c_1 in club network g_2, and thus end up better off. In particular, in the second round of moves, agents i_1 and i_2 leave club c_1 and move to club c_2 - changing club network g_1 to club network g_2. This move makes agents i_1 and i_2 better off, but also makes agents i_6 and i_7 better off. In particular, agents i_1 and i_2 move from club c_1 in network g_1 with 5 members $\{i_1, i_2, i_3, i_6, i_7\}$ and payoffs given by

$$u(\left|g_1^2(i_1)\right|) = u(\left|g_1^2(i_2)\right|) = u(|\{i_1, i_2, i_3, i_6, i_7\}|) = u(5),$$

to club c_2 in network g_2 with 4 members $\{i_1, i_2, i_4, i_5\}$ and payoffs given by

$$u(\left|g_2^2(i_1)\right|) = u(\left|g_2^2(i_2)\right|) = u(|\{i_1, i_2, i_4, i_5\}|) = u(4).$$

These second round moves by agents i_1 and i_2 leave agents i_6 and i_7 in a smaller club c_1 and thus make agents i_6 and i_7 better off. Thus, agents i_6 and i_7 who started out in club c_2 in network g_0 with 4 members $\{i_4, i_5, i_6, i_7\}$ and payoffs given by

$$u(\left|g_0^2(i_6)\right|) = u(\left|g_0^2(i_7)\right|) = u(|\{i_4, i_5, i_6, i_7\}|) = u(4),$$

end up in club c_1 in network g_2 with 3 members, $\{i_3, i_6, i_7\}$ and payoffs given by

$$u(\left|g_2^2(i_6)\right|) = u(\left|g_2^2(i_7)\right|) = u(|\{i_3, i_6, i_7\}|) = u(3).$$

[6] Allowing coalitions to initially be made worse off but then eventually better off, as in this example, differentiates farsighted dominance from other dominance relations.

3.2 Path Dominance

We say that a sequence of club networks $\{g_k\}_k$ in $\mathbb{K}$ forms a *farsighted domination path* (i.e., a $\lhd\lhd$-path) if for any two consecutive networks g_{k-1} and g_k,

$$\begin{gathered} g_k \text{ farsightedly dominates } g_{k-1}, \\ \text{that is,} \\ \text{if for any two consecutive networks } g_{k-1} \text{ and } g_k, \\ g_{k-1} \lhd\lhd g_k. \end{gathered}$$

Using the terminology of graph theory, we can think of the farsighted dominance relation $g_{k-1} \lhd\lhd g_k$ between club networks g_k and g_{k-1} as defining a $\lhd\lhd$-arc *from* network g_{k-1} *to* network g_k. The *length* of $\lhd\lhd$-path $\{g_k\}_k$ is defined to be the number of $\lhd\lhd$-arcs in the path. We say that network $g_1 \in \mathbb{K}$ is $\lhd\lhd$-*reachable* from network $g_0 \in \mathbb{K}$ if there exists a *finite* $\lhd\lhd$-path *in* $\mathbb{K}$ from g_0 to g_1.

We can use the notion of $\lhd\lhd$-reachability to define a new relation on the feasible set of club networks $\mathbb{K}$. In particular, for any two networks g_0 and g_1 in $\mathbb{K}$ define

$$g_1 \trianglerighteq_{\mathbb{K}} g_0 \text{ if and only if } \begin{cases} g_1 \text{ is } \lhd\lhd \text{-reachable from } g_0 \text{ through } \mathbb{K} \text{ , or} \\ g_1 = g_0. \end{cases} \tag{1}$$

The relation $\trianglerighteq_{\mathbb{K}}$ is a weak ordering on $\mathbb{K}$. In particular, $\trianglerighteq_{\mathbb{K}}$ is reflexive ($g \trianglerighteq_{\mathbb{K}} g$) and $\trianglerighteq_{\mathbb{K}}$ is transitive ($g_2 \trianglerighteq_{\mathbb{K}} g_1$ and $g_1 \trianglerighteq_{\mathbb{K}} g_0$ implies that $g_2 \trianglerighteq_{\mathbb{K}} g_0$). We shall refer to the relation $\trianglerighteq_{\mathbb{K}}$ as the *farsighted domination path (FDP) relation.*[7]

Note that if club network g_1 is a feasible improvement over club network g_0 for agents $i \in S$, then g_1 also dominates g_0 with respect to the farsighted domination path (FDP) relation, $\trianglerighteq_{\mathbb{K}}$. Thus,

$$\text{if } g_1 \rhd_S g_0 \text{ for some coalition } S, \text{ then } g_1 \trianglerighteq_{\mathbb{K}} g_0.$$

This applies even if the S consists of a single agent, that is, even if $S = \{i\}$ for some agent $i \in N$. Thus,

$$\text{if } g_1 \rhd_{\{i\}} g_0 \text{ for some agent } i \in N, \text{ then } g_1 \trianglerighteq_{\mathbb{K}} g_0.$$

Remark 1 *If network $g_0 \in \mathbb{K}$ is $\lhd\lhd$-reachable from network g_0, then we say that $\mathbb{K}$ contains a $\lhd\lhd$-*circuit*. Thus, a $\lhd\lhd$-circuit in $\mathbb{K}$ starting at club network $g_0 \in \mathbb{K}$ is a finite $\lhd\lhd$-path from g_0 to g_0. A $\lhd\lhd$-circuit of length 1 is called a $\lhd\lhd$-loop. Note that because the relation $\lhd\lhd$ is irreflexive (i.e., because it is not possible to have $g \lhd\lhd g$) $\lhd\lhd$-loops are in fact ruled out. However, because the farsighted dominance relation, $\lhd\lhd$, is* not transitive*, it is possible to have $\lhd\lhd$-circuits of length greater than 1.*

[7] The relation $\trianglerighteq_{\mathbb{K}}$ is sometimes referred to as the transitive closure in $\mathbb{K}$ of the farsighted dominance relation, $\lhd\lhd$, on $\mathbb{K}$.

4 Club Formation Games and the Farsighted Core

A club formation game with farsighted agents is a pair $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$, where $\mathbb{K}$ is the feasible set of club networks and $\trianglerighteq_{\mathbb{K}}$ is the farsighted domination path (FDP) relation on $\mathbb{K}$.

One of the most fundamental stability notions in game theory is the core. Here we define the notion of core for club formation games with respect to farsighted path dominance. We call this notion of the core the *farsighted core*.

Definition 3 *(The Farsighted Core)*

Let $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ be a farsighted club formation game. A subset $\mathbb{C}$ of club networks in $\mathbb{K}$ is said to be the farsighted core of $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ if for each club network $g \in \mathbb{C}$ there does not exist a club network $g' \in \mathbb{K}$, $g' \neq g$, such that $g' \trianglerighteq_{\mathbb{K}} g$.

Note that any club network g contained in the farsighted core $\mathbb{C}$ is a Nash club network - and in fact is a strong Nash club network.[8] Letting $\mathbb{NE}$ denote the set of Nash club networks in $\mathbb{K}$ and letting $\mathbb{SNE}$ denote the set of strong Nash club networks in $\mathbb{K}$, we can conclude from our definition of the farsighted core that

$$\mathbb{C} \subseteq \mathbb{SNE} \subseteq \mathbb{NE}.$$

Example 3 is particularly interesting as it demonstrates that farsighted behavior may generate quite different outcomes than myopic behavior and strong Nash equilibria (or Nash club equilibria). In Example 3, the number of clubs is not sufficiently large to permit all players to be in clubs of optimal size (i.e., $|C| < \frac{|N|}{s^*}$ for $|C| = 2$, $|N| = 7$, and $s^* = 3$). As shown in Arnold and Wooders (2002), in this case, it is a strong Nash equilibrium for the agents to be divided into clubs that are as close as possible to the same size – in this example, into clubs of sizes 3 and 4. No group of agents (nor any single agent) can improve upon his own payoff - but, nevertheless, the farsighted core is empty. This is because, as the example illustrates, farsighted agents, unlike myopic agents, will switch their club memberships to an already overcrowded club, temporarily making themselves worse off, if in the end switching induces an out migration that makes them better off.

When the number of clubs is unconstraining, the situation is quite different. Our next results give necessary and sufficient conditions for the farsighted core of a club formation game to be nonempty when there is an ample number of clubs, that is, when the number of clubs is unconstraining.

Theorem 1 *(Necessary and sufficient conditions for nonemptiness of the farsighted core)*

Consider a farsighted club formation game $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ with N agents, C clubs, and optimal club size s^*, $1 \leq s^* < |N|$. Suppose that assumptions (A-1)-(A-4) hold. In addition, assume that

[8] A club network $g \in \mathbb{K}$, is a Nash club network if there does not exist another club network $g' \in \mathbb{K}$ such that $g' \triangleright_{\{i\}} g$ for some agent $i \in N$.

A club network $g \in \mathbb{K}$, is a strong Nash club network if there does not exist another club network $g' \in \mathbb{K}$ such that $g' \triangleright_S g$ for some coalition S.

(a) $|C| \geq \frac{|N|}{s^*}$*, and*

(b) $|N| = rs^* + l$ *for nonnegative integers* r *and* l*,* $l < s^*$*.*

The following statements are true.

1. *The farsighted core of* $(\mathbb{K}, \trianglerighteq_{\mathbb{K}})$ *is nonempty if and only if* $u(l) \geq u(s^* + 1)$*.*
2. *Club network* g_* *is contained in the farsighted core if and only if* g_* *has* r *clubs of size* s^* *and one club of size* l*.*

Proof. Suppose that

$$|C| \geq \frac{|N|}{s^*} \text{ and } u(l) \geq u(s^* + 1).$$

Consider a club network g_* with r clubs of size s^* and one club of size l ($l < s^*$). Let I be the group of agents such that each agent i in I is a member of as s^* club (i.e., a club of size s^*) and let E be the group of agents in the club of size l. Because

$$u(\left|g_*^2(i)\right|) \geq u(\left|g^2(i)\right|) \text{ for all } g \in \mathbb{K} \text{ and all } i \in I,$$

no coalition requiring the participation of agents from I will be able to initiate a change in club network g_* which leads to another club network making the participates from I better off. Moreover, because

$$u(l) \geq u(s^* + 1) \text{ and payoffs are single peaked,}$$

no coalition of agents from E alone will be able to initiate a change in club network g_* which leads to another club network making the agents from E better off. Thus, for any club network g_* with r clubs of size s^* and one club of size l, there does not exist a club network $g \in \mathbb{K}$, $g \neq g_*$, such that $g \trianglerighteq_{\mathbb{K}} g_*$. Therefore, if $|C| \geq \frac{|N|}{s^*}$ and $u(l) \geq u(s^* + 1)$, then any club network g_* with r clubs of size s^* and one club of size l is in the farsighted core.

Suppose now that $|C| \geq \frac{|N|}{s^*}$ but that $u(l) < u(s^* + 1)$. Let $g \in \mathbb{K}$ and given g define the following club subcollections:

$$C_g^+ := \{c \in C : |g(c)| > s^*\},$$

$$C_g^* := \{c \in C : |g(c)| = s^*\},$$
and
$$C_g^- := \{c \in C : |g(c)| < s^*\}$$

Given that $|C| \geq \frac{|N|}{s^*}$, $C_g^- \neq \emptyset$ for all $g \in \mathbb{K}$.

Let $g \in \mathbb{K}$ and suppose that $C_g^+ \neq \emptyset$. Consider clubs $c_1 \in C_g^+$ and $c_2 \in C_g^-$ and let S_1 be a coalition of agents from club c_1 of size $s^* - |g(c_2)|$. Observe that if agents in coalition $S_1 \subseteq g(c_1)$ switch their memberships to club c_2, then the new larger club c_2 will be of optimal size s^* and all members of coalition S_1 will be made better off by making the switch. Let $g' \in \mathbb{K}$ be the club network which results from this switch. Then we have

$$g' \triangleright_{S_1} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g.$$

Let $g \in \mathbb{K}$ and suppose that $C_g^+ = \emptyset$. If $\left|C_g^*\right| = r$, then there is an agent i in some club $c_1 \in C_g^-$ who can switch his membership to some club $c_2 \in C_g^*$ and be made better off because $u(l) < u(s^* + 1)$. Letting $g' \in \mathbb{K}$ be the club network resulting from this switch we have

$$g' \triangleright_{\{i\}} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g.$$

If $\left|C_g^*\right| < r$ (maintaining he assumption that $C_g^+ = \emptyset$) then sufficiently many agents from clubs in C_g^- can switch their memberships to some club $c' \in C_g^-$ resulting in a new, larger club c' of optimal size s^*. Moreover, all agents making this membership switch will be better off. Letting S' denote the coalition of agents making the switch and letting $g' \in \mathbb{K}$ be the resulting club network we have

$$g' \triangleright_{S'} g \text{ and thus } g' \trianglerighteq_{\mathbb{K}} g.$$

■

5 Conclusions

An aspect of our work which we find particularly interesting is relationships between the outcomes of the dynamic process in Arnold and Wooders (2002) and the outcomes of farsighted strategic behavior. Research in progress addresses these questions.

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